FINITE W-SUPERALGEBRAS AND TRUNCATED SUPER YANGIANS

YUNG-NING PENG

Abstract. We introduce the super Yangian $Y^{\mathfrak{b}}_{m|n}$ of general linear Lie superalgebra associated to any Borel subalgebra $\mathfrak{b}$ for $\mathfrak{gl}_{m|n}$. Let $e \in \mathfrak{gl}_{ml|nl}$ be a "rectangular" nilpotent element and $\mathcal{W}_e$ be the associated finite W-superalgebra. We show that, for each $\mathfrak{b}$, the truncated super Yangian $Y^{\mathfrak{b},l}_{m|n}$ of level l is isomorphic to $\mathcal{W}_e$. As a consequence, the super Yangians $Y^{\mathfrak{b}}_{m|n}$ are shown to be isomorphic for all $\mathfrak{b}$.

1. Introduction

The connection between Yangians and finite W-algebras of type A was first noticed by mathematical physicists Ragoucy and Sorba in [RS] in some special cases, and then constructed in general cases by Brundan and Kleshchev explicitly in [BK]. In this way, a realization of finite W-algebra in terms of truncated Yangian is obtained, and this provides a useful tool for the study of the representation theory of finite W-algebras.

In this note, we build such a connection between finite W-superalgebras and super Yangians of type A where the nilpotent element e is *rectangular* (cf. §3 for the precise definition). The Borel subalgebras are no longer all conjugate by the corresponding Weyl group in the general linear Lie superalgebra $\mathfrak{gl}_{m|n}$.

Let $\mathfrak{b}$ be a fixed ϵ-δ sequence of $\mathfrak{gl}_{m|n}$ (cf. [CW]). The set of such sequences is in one-to-one correspondence with the set of the Borel subalgebras of $\mathfrak{gl}_{m|n}$ up to Weyl group conjugation. For each $\mathfrak{b}$ and a non-negative integer l, we define $Y^{\mathfrak{b}}_{m|n}$, the super Yangian for $\mathfrak{gl}_{m|n}$ associated to $\mathfrak{b}$, and the truncated super Yangian of level l, denoted by $Y^{\mathfrak{b},l}_{m|n}$, as a quotient of $Y^{\mathfrak{b}}_{m|n}$.

The original definition of super Yangian in [Na] corresponds to the *standard Borel* case, where all the δ's appear before all the ϵ's. It is non-trivial from definition that whether or not $Y^{\mathfrak{b}}_{m|n}$ and $Y^{\mathfrak{b}'}_{m|n}$ are isomorphic when $\mathfrak{b}$ and $\mathfrak{b}'$ are distinct.

Our main result is that for any fixed ϵ-δ sequence $\mathfrak{b}$, there exists an isomorphism of filtered algebras between $Y^{\mathfrak{b},l}_{m|n}$ and $\mathcal{W}_e$, the finite W-superalgebra associated to a rectangular e in $\mathfrak{gl}_{ml|nl}$. When the Borel is the standard one, such a connection was firstly obtained in [BR] in a different approach. In this article, we provide a new proof which applies to any $\mathfrak{b}$. Our proof is similar to [BK]. In particular, the specialization of our results when $m = n = 1$ coincides with the rectangular shape case of [BBG].

An interesting observation is that $\mathcal{W}_e$ is independent of the choices of $\mathfrak{b}$. Indeed, choosing different $\mathfrak{b}$ means using different basis to describe the same Levi subalgebra of $\mathfrak{gl}_{ml|nl}$. As a result, we proved that the super Yangians $Y_{m|n}^{\mathfrak{b}}$ are all isomorphic for every $\mathfrak{b}$.

2. Super Yangians $Y_{m|n}^{\mathfrak{b}}$

Let m and n be non-negative integers. Let $\mathfrak{b}$ be an ϵ-δ sequence of $\mathfrak{gl}_{m|n}$ introduced in [CW]. That is, $\mathfrak{b}$ is a sequence consisting of exactly m δ's and n ϵ's, both indistinguishable. Note that the set of ϵ-δ sequence of $\mathfrak{gl}_{m|n}$ is in one-to-one correspondence with the set of the Borel subalgebras of $\mathfrak{gl}_{m|n}$ up to Weyl group conjugation.

For each $1 \leq i \leq m+n$, define a number $|i| \in \mathbb{Z}_2$, called the parity of i, as follows:

$$|i| := \begin{cases} \overline{0} & \text{if the } i\text{-th position of } \mathfrak{b} \text{ is } \delta, \\ \overline{1} & \text{if the } i\text{-th position of } \mathfrak{b} \text{ is } \epsilon. \end{cases} \tag{2.1}$$

For a given $\mathfrak{b}$, the super Yangian $Y_{m|n}^{\mathfrak{b}}$ is the associative $\mathbb{Z}_2$-graded algebra (i.e., superalgebra) over $\mathbb{C}$ with generators

$$\left\{ t_{ij;\mathfrak{b}}^{(r)} \mid 1 \leq i,j \leq m+n; r \geq 0 \right\}, \tag{2.2}$$

where $t_{ij;\mathfrak{b}}^{(0)} := \delta_{ij}$ and defining relations

$$[t_{ij;\mathfrak{b}}^{(r)}, t_{hk;\mathfrak{b}}^{(s)}] = (-1)^{|i|\,|j|+|i|\,|h|+|j|\,|h|} \sum_{t=0}^{\min(r,s)-1} \left(t_{hj;\mathfrak{b}}^{(t)} t_{ik;\mathfrak{b}}^{(r+s-1-t)} - t_{hj;\mathfrak{b}}^{(r+s-1-t)} t_{ik;\mathfrak{b}}^{(t)} \right), \tag{2.3}$$

where the bracket is understood as a supercommutator.

For $r > 0$, the element $t_{ij;\mathfrak{b}}^{(r)}$ is defined to be an odd element if $|i|+|j| = \overline{1}$ and an even element if $|i|+|j| = \overline{0}$. In the case when $m=0$ or $n=0$, it reduces to the usual Yangian. The generators in (2.2) are called the RTT generators while the relations (2.3) are called the RTT relations.

It is non-trivial from the definition that whether or not $Y_{m|n}^{\mathfrak{b}}$ and $Y_{m|n}^{\mathfrak{b}'}$ are isomorphic when $\mathfrak{b}$ and $\mathfrak{b}'$ are different. For convenience, in this article we will fix such a $\mathfrak{b}$ and when it's suitable we omit the $\mathfrak{b}$ in our notation; that is, $Y_{m|n} = Y_{m|n}^{\mathfrak{b}}$ and $t_{ij}^{(r)} = t_{ij;\mathfrak{b}}^{(r)}$. Note that the original definition in [Na] corresponds to the *standard Borel case*, which means $\mathfrak{b} = \overbrace{\delta \cdots \delta}^{m} \overbrace{\epsilon \cdots \epsilon}^{n}$.

For all $1 \leq i,j \leq m+n$, we define the formal power sequence

$$t_{ij}(u) := \sum_{r \geq 0} t_{ij}^{(r)} u^{-r}.$$

It is well-known (cf. [Go]) that $Y_{m|n}$ is a Hopf-algebra, where the comultiplication $\Delta : Y_{m|n} \to Y_{m|n} \otimes Y_{m|n}$ is defined by

$$\Delta(t_{ij}^{(r)}) = \sum_{s=0}^{r} \sum_{k=1}^{m+n} t_{ik}^{(r-s)} t_{kj}^{(s)}, \tag{2.4}$$

and one has the evaluation homomorphism $\mathrm{ev} : Y_{m|n} \to U(\mathfrak{gl}_{m|n})$ defined by

$$\mathrm{ev}\left(t_{ij}(u)\right) := \delta_{ij} + (-1)^{|i|} e_{i,j}, \tag{2.5}$$

where $e_{i,j}$ denotes the elementary matrix.

Definition 2.1. Let l be a non-negative integer and $I_l^{\mathfrak{b}}$ be the 2-sided ideal of $Y_{m|n}^{\mathfrak{b}}$ generated by the elements $\{t_{ij;\mathfrak{b}}^{(r)} | 1 \leq i,j \leq m+n, r > l\}$. The *truncated super Yangian of level* l, denoted by $Y_{m|n}^{\mathfrak{b},l}$, is defined to be the quotient $Y_{m|n}^{\mathfrak{b}}/I_l^{\mathfrak{b}}$.

Again, for the sake of convenience, when it's appropriate we will omit the fixed $\mathfrak{b}$ in our notations so that $I_l = I_l^{\mathfrak{b}}$ and $Y_{m|n}^l = Y_{m|n}^{\mathfrak{b},l}$.

Equivalently, $Y_{m|n}^l$ can be described as the superalgebra generated by the RTT generators (2.2) subjects to the RTT relations (2.3) plus the following relations:

$$t_{ij}^{(r)} = 0, \text{ for all } 1 \leq i,j \leq m+n \text{ and } r > l.$$

In fact, there's one more way to describe $Y_{m|n}^l$. We define the following homomorphism

$$\kappa_l := (\overbrace{\mathrm{ev} \otimes \cdots \otimes \mathrm{ev}}^{l-copies}) \circ \Delta^l : Y_{m|n} \to U(\mathfrak{gl}_{m|n})^{\otimes l}, \tag{2.6}$$

then we have

$$\kappa_l(t_{ij}^{(r)}) = \sum_{1 \leq s_1 < \cdots < s_r \leq l} \; \sum_{1 \leq i_1, \ldots, i_{r-1} \leq m+n} (-1)^{|i|+|i_1|+\cdots+|i_{r-1}|} e_{i,i_1}^{[s_1]} e_{i_1,i_2}^{[s_2]} \cdots e_{i_{r-1},j}^{[s_r]}, \tag{2.7}$$

where $e_{i,j}^{[s]} := 1^{\otimes(s-1)} \otimes e_{i,j} \otimes 1^{\otimes(l-s)}$.

One may observe that the kernel of κ_l is exactly the 2-sided ideal I_l. As a consequence, we may identify $Y_{m|n}^l$ with the image of $Y_{m|n}$ under the map κ_l. In particular, the induced map

$$\kappa_l : Y_{m|n}^l \to U(\mathfrak{gl}_{m|n})^{\otimes l} \tag{2.8}$$

is an injective homomorphism.

It should be clear from the context that we are dealing with $Y_{m|n}$ or $Y_{m|n}^l$ hence we will use the same notation $t_{ij}^{(r)}$ to denote its image in $Y_{m|n}^l$ by abusing notations.

The following theorem is a PBW theorem for $Y_{m|n}^{\mathfrak{b}}$ and $Y_{m|n}^{\mathfrak{b},l}$. We remark here that the proof of [Go, Theorem 1] applies in our setting for any fixed $\mathfrak{b}$.

Proposition 2.2. *Let $\mathfrak{b}$ be fixed. Then the set of all supermonomials in the elements*

$$\left\{t_{ij;\mathfrak{b}}^{(r)} | 1 \leq i,j \leq m+n, 1 \leq r \leq l\right\}$$

taken in some fixed order (containing no second or higher order powers of the odd generators) forms a basis for $Y^{\mathfrak{b},l}_{m|n}$.

Similarly, the set of all supermonomials in the elements

$$\left\{t^{(r)}_{ij;\mathfrak{b}} | 1 \leq i, j \leq m+n, 1 \leq r\right\}$$

taken in some fixed order (containing no second or higher order powers of the odd generators) forms a basis for $Y^{\mathfrak{b}}_{m|n}$.

Define the canonical filtration of $Y_{m|n}$

$$F_0 Y_{m|n} \subseteq F_1 Y_{m|n} \subset F_2 Y_{m|n} \subseteq \cdots$$

by $\deg t^{(r)}_{ij} := r$, i.e., $F_d Y_{m|n}$ is the span of all supermonomials in the generators $t^{(r)}_{ij}$ of total degree $\leq$ d. It is clear form (2.3) that the associated graded algebra $\operatorname{gr} Y_{m|n}$ is supercommutative. We also obtain the canonical filtration on $Y^l_{m|n}$ induced from the natural quotient map $Y_{m|n} \twoheadrightarrow Y^l_{m|n}$.

3. Finite W-superalgebras

Let M and N be non-negative integers. Let $\mathfrak{g} = \mathfrak{gl}_{M|N}$ and $(\cdot\,,\cdot)$ denote the non-degenerate even supersymmetric invariant bilinear form on $\mathfrak{g}$ defined by

$$(x, y) := str(xy), \forall x, y \in \mathfrak{g},$$

where str means the super trace and xy stands for the usual composition. Every elements of $\mathfrak{g}$ in the context are considered $\mathbb{Z}_2$-homogeneous unless mentioned specifically.

Let e be an even nilpotent element in $\mathfrak{g}$. It can be shown (cf. [Ho], [Wa]) that there exists (not uniquely) a semisimple element $h \in \mathfrak{g}$ such that $\operatorname{ad} h : \mathfrak{g} \to \mathfrak{g}$ gives a *good* $\mathbb{Z}$*-grading of* $\mathfrak{g}$ *for* e, which means the following conditions are satisfied:

(1) $\operatorname{ad} h(e) = 2e$,
(2) $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j)$, where $\mathfrak{g}(j) := \{x \in \mathfrak{g} | \operatorname{ad} h(x) = jx\}$,
(3) the center of $\mathfrak{g}$ in contained in $\mathfrak{g}(0)$,
(4) $\operatorname{ad} e : \mathfrak{g}(j) \to \mathfrak{g}(j+2)$ is injective for all $j \leq -1$,
(5) $\operatorname{ad} e : \mathfrak{g}(j) \to \mathfrak{g}(j+2)$ is surjective for all $j \geq -1$.

In this article we only care abut the case where the grading is *even*, that is, we always assume that $\mathfrak{g}(i) = 0$ for all $i \notin 2\mathbb{Z}$.

Define the nilpotent subalgebras of $\mathfrak{g}$ as follows:

$$\mathfrak{p} := \bigoplus_{j \geq 0} \mathfrak{g}(j), \quad \mathfrak{m} := \bigoplus_{j < 0} \mathfrak{g}(j). \tag{3.1}$$

Let $\chi \in \mathfrak{g}^*$ be defined by $\chi(y) := (y, e)$, for all $y \in \mathfrak{g}$. Then the restriction of χ on $\mathfrak{m}$ gives a one dimensional $U(\mathfrak{m})$-module. Let I_χ denote the left ideal of $U(\mathfrak{g})$ generated by $\{a - \chi(a) | a \in \mathfrak{m}\}$. By PBW theorem of $U(\mathfrak{g})$, we have $U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I_\chi$. Let $\mathrm{pr}_\chi : U(\mathfrak{g}) \to U(\mathfrak{p})$ be the natural projection and we can

identify $U(\mathfrak{g})/I_\chi \cong U(\mathfrak{p})$. Furthermore we define the χ-twisted action of $\mathfrak{m}$ on $U(\mathfrak{p})$ by

$$a \cdot y := \mathrm{pr}_\chi([a, y]) \text{ for all } a \in \mathfrak{m} \text{ and } y \in U(\mathfrak{p}).$$

The *finite* W-*superalgebra* (which we will omit the term "finite" from now on) is defined to be the $\mathfrak{m}$-invariant space of $U(\mathfrak{p})$ under the χ-twisted action; that is,

$$\begin{aligned}\mathcal{W}_{e,h} := U(\mathfrak{p})^{\mathfrak{m}} =& \{y \in U(\mathfrak{p}) \,|\, \mathrm{pr}_\chi([a, y]) = 0, \forall a \in \mathfrak{m}\}\\ =& \{y \in U(\mathfrak{p}) \,|\, \big(a - \chi(a)\big) y \in I_\chi, \forall a \in \mathfrak{m}\}.\end{aligned}$$

At this point, the definition of W-superalgebra depends on the nilpotent element e and a semisimple element h which gives a good $\mathbb{Z}$-grading for e.

Example 3.1. If we take the nilpotent element $e = 0$, then $\chi = 0$, $\mathfrak{g} = \mathfrak{g}(0) = \mathfrak{p}$, $\mathfrak{m} = 0$, $\mathcal{W}_{e,h} = U(\mathfrak{p}) = U(\mathfrak{g})$.

Now we introduce certain combinatorial objects called (m, n)-*colored rectangles* (which are in fact special cases of the so called *pyramids*). These objects provide a diagrammatic way to record the information needed to define W-superalgebras.

Let π be a rectangular Young diagram with $m+n$ boxes as its height and l boxes as its base. We choose arbitrary m rows and color the boxes in these rows by $+$, while we color the other n rows by $-$. Such a diagram is called an *(m,n)-colored rectangle*, or a rectangle for short. For example,

$$\pi = \begin{array}{|c|c|c|c|}\hline + & + & + & + \\ \hline - & - & - & - \\ \hline + & + & + & + \\ \hline - & - & - & - \\ \hline - & - & - & - \\ \hline\end{array}$$

Set $M = ml$ and $N = nl$. We enumerate the $\boxed{+}$ boxes by the numbers $\{\overline{1}, \ldots, \overline{M}\}$ down columns from left to right, and enumerate the $\boxed{-}$ boxes by the numbers $\{1, \ldots, N\}$ in the same fashion. In fact, we may enumerate the boxes by an arbitrary order as long as the parities are preserved so we just choose the easiest way according to our purpose. Moreover, we image that each box of π is of size 2×2 and π is built on the x-axis where the center of π is exactly on the origin. For example,

$$\pi = \begin{array}{|c|c|c|c|}\hline \overline{1} & \overline{3} & \overline{5} & \overline{7} \\ \hline 1 & 4 & 7 & 10 \\ \hline \overline{2} & \overline{4} & \overline{6} & \overline{8} \\ \hline 2 & 5 & 8 & 11 \\ \hline 3 & 6 & 9 & 12 \\ \hline \multicolumn{4}{c}{} \\ -3 & -1 & 1 & 3 \end{array} \tag{3.2}$$

We now explain how to read off an even nilpotent $e(\pi)$ and a semisimple $h(\pi)$ in $\mathfrak{g} = \mathfrak{gl}_{M|N}$ which gives a good $\mathbb{Z}$-grading of $\mathfrak{g}$ for e from a given rectangle π. Let $J = \{\overline{1} < \ldots < \overline{M} < 1 < \ldots < N\}$ be an ordered index set and let $\{e_i | i \in J\}$ be a basis of $\mathbb{C}^{M|N}$. We identify $\mathfrak{gl}_{M|N} \cong \mathrm{End}\,(\mathbb{C}^{M|N})$ with the $(M+N) \times (M+N)$ matrices over $\mathbb{C}$ by this basis of $\mathbb{C}^{M|N}$ with respect to the order $e_i < e_j$ if $i < j$ in J.

Define the element

$$e(\pi) := \sum_{i,j \in J} e_{i,j} \in \mathfrak{gl}_{M|N}, \tag{3.3}$$

summing over all adjacent pairs $\boxed{i}\boxed{j}$ of boxes in π. It is clear that such an element $e(\pi)$ is even nilpotent.

Let $\tilde{\mathrm{col}}(i)$ denote the x-coordinate of the box numbered with $i \in J$. For instance, in our example (3.2), $\tilde{\mathrm{col}}(\overline{1}) = -3$ and $\tilde{\mathrm{col}}(8) = 1$. Then we define the following diagonal matrix

$$h(\pi) := -\mathrm{diag}\big(\tilde{\mathrm{col}}(\overline{1}), \ldots, \tilde{\mathrm{col}}(\overline{M}), \tilde{\mathrm{col}}(1), \ldots, \tilde{\mathrm{col}}(N)\big) \in \mathfrak{g}. \tag{3.4}$$

One may check directly that $\mathrm{ad}\, h(\pi)$ gives a good $\mathbb{Z}$-grading of $\mathfrak{g}$ for $e(\pi)$.

Remark 3.2. In general, there are other even good $\mathbb{Z}$-gradings for e. However, if our $e = e(\pi)$ is obtained by a rectangle π according to (3.3), then such a grading is unique by [Ho, Theorem 7.2] because we can't shift the rows of π to obtain pyramids other than itself.

Now we characterize those e obtained from (3.3) for some rectangle π. Consider

$$e = e_M \oplus e_N \in \mathrm{End}\,\mathbb{C}^{M|N}, \tag{3.5}$$

where e_M and e_N are the restriction of e to $\mathbb{C}^{M|0}$ and $\mathbb{C}^{0|N}$, respectively. Let $\mu = (\mu_1, \mu_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots)$ be the partitions representing the Jordan type of e_M and e_N, respectively.

Definition 3.3. An element e is called *rectangular* if it is even nilpotent and the Jordan blocks of e_M and e_N are all of the same size l, i.e., $\mu = (\overbrace{l, \ldots, l}^{m-copies})$ and $\nu = (\overbrace{l, \ldots, l}^{n-copies})$ for some for some non-negative integers l, m, n.

Clearly, e is of the form (3.3) for some rectangle π if and only if e is rectangular. Assume now e is rectangular. We define a new partition λ by collecting all parts of μ and ν together and reorder them by an arbitrary order. Since all the parts are the same number l, we use $\overline{l}$ to denote the parts obtained from μ.

For example, consider $\mathfrak{gl}_{8|12}$, $\mu = (\overline{4}, \overline{4})$ and $\nu = (4, 4, 4)$. Then one possible λ is $\lambda = (\overline{4}, 4, \overline{4}, 4, 4)$.

Next we read off an ϵ-δ sequence $\mathfrak{b}$ from λ : if the i-th position of λ is l (respectively, $\overline{l}$), then the i-th position of $\mathfrak{b}$ is ϵ (respectively, δ). For example, the λ above corresponds to the ϵ-δ sequence $\mathfrak{b} = \delta\epsilon\delta\epsilon\epsilon$.

Then we color the rectangle of height $m+n$ base l with respect to $\mathfrak{b}$: we color the i-th row of π by $+$ (respectively, $-$) if the i-th position of $\mathfrak{b}$ is δ (respectively, ϵ) where the rows are counted from top to bottom. After coloring the rows, we enumerate the boxes of π by exactly the same fashion explained in the paragraph before (3.2).

Therefore, we have a bijection between the set of (m,n)-colored rectangles of base l and the set of pairs $(e,\mathfrak{b})$ where e is a rectangular element in $\mathfrak{gl}_{ml|nl}$ and $\mathfrak{b}$ is an ϵ-δ sequence containing exactly m δ's and n ϵ's.

Let π be a fixed (m,n)-colored rectangle and $e(\pi)$ denote the rectangular element corresponds to π. We will denote by $\mathcal{W}_\pi := \mathcal{W}_{e(\pi)}$ the W-superalgebra associated to $e(\pi)$. Note that we may omit $h(\pi)$ in our notation by Remark 3.2 without causing any ambiguity.

Remark 3.4. Another interesting observation is that $\mathcal{W}_\pi$ is independent of the choices of the sequence $\mathfrak{b}$ because any other sequence $\mathfrak{b}'$ yields to the same e and hence the same W-superalgebra.

Now we label the columns of π from left to right by $1,\dots,l$, and for any $i\in J$ we let $\mathrm{col}(i)$ denote the column where i appear. Define the *Kazhdan filtration* of $U(\mathfrak{gl}_{M|N})$

$$\cdots \subseteq F_dU(\mathfrak{gl}_{M|N}) \subseteq F_{d+1}U(\mathfrak{gl}_{M|N}) \subseteq \cdots$$

by declaring

$$\deg(e_{i,j}) := \mathrm{col}(j) - \mathrm{col}(i) + 1 \tag{3.6}$$

for each $i,j\in J$ and $F_dU(\mathfrak{gl}_{M|N})$ is the span of all supermonomials $e_{i_1,j_1}\cdots e_{i_s,j_s}$ for $s\geq 0$ and $\sum_{k=1}^{s}\deg\,(e_{i_k,j_k})\leq d$. Let $\mathrm{gr}\,U(\mathfrak{gl}_{M|N})$ denote the associated graded superalgebra. The natural projection $\mathfrak{gl}_{M|N}\twoheadrightarrow\mathfrak{p}$ induces a grading on $\mathcal{W}_\pi$.

On the other hand, let $\mathfrak{g}^e$ denote the centralizer of e in $\mathfrak{g}=\mathfrak{gl}_{M|N}$ and let $S(\mathfrak{g}^e)$ denote the associated supersymmetric algebra. We define the Kazhdan filtration on $S(\mathfrak{g}^e)$ by the same setting (3.6). The following proposition was observed in [Zh], where the mild assumption on e there is satisfied when e is rectangular.

Proposition 3.5. [Zh, Remark 3.9] $\mathrm{gr}\,\mathcal{W}_\pi\cong S(\mathfrak{g}^e)$ *as Kazhdan graded superalgebras.*

The following proposition is a well-known result about $\mathfrak{g}^e$. As remarked in [BBG], the result is similar to the Lie algebra case $\mathfrak{gl}_{M+N}$ because e is even.

Proposition 3.6. *Let π be an (m,n)-colored rectangle and $e=e(\pi)$ be the associated rectangular nilpotent element. For all $1\leq i,j\leq m+n$ and $r>0$, define*

$$c_{i,j}^{(r)} := \sum_{\substack{1\leq h,k\leq m+n\\ row(h)=i,row(k)=j\\ col(k)-col(h)=r-1}} (-1)^{|i|}e_{h,k}\in\mathfrak{g}=\mathfrak{gl}_{M|N}.$$

Then the set of vectors $\{c_{i,j}^{(r)}|1\leq i,j\leq m+n, 1\leq r\leq l\}$ forms a basis for $\mathfrak{g}^e$.

Corollary 3.7. *Let $\mathfrak{b}$ be fixed. Consider $Y_{m|n}^{\mathfrak{b},l}$ with the canonical filtration and $S(\mathfrak{g}^e)$ with the Kazhdan filtration. Let $F_d Y_{m|n}^{\mathfrak{b},l}$ and $F_d S(\mathfrak{g}^e)$ denote the associated filtered algebras, respectively. Then for each $d \geq 0$, we have $\dim F_d Y_{m|n}^{\mathfrak{b},l} = \dim F_d S(\mathfrak{g}^e)$.*

Proof. Follows from Proposition 2.2, Proposition 3.6 and induction on d. □

4. Isomorphism between $Y_{m|n}^{\mathfrak{b},l}$ and $\mathcal{W}_\pi$

Let π be a given (m,n)-colored rectangle with base l and $\mathfrak{b}$ be the ϵ-δ sequence determined by the colors of rows of π. We now define some elements in $U(\mathfrak{p})$. It turns out later that they are $\mathfrak{m}$-invariant, i.e., belong to $\mathcal{W}_\pi$.

For each $1 \leq r \leq l$, define

$$\rho_r := -(l-r)(m-n). \tag{4.1}$$

For $1 \leq i,j \leq m+n$, define

$$\tilde{e}_{i,j} := (-1)^{\mathrm{col}(j)-\mathrm{col}(i)}(e_{i,j} + \delta_{ij}(-1)^{|i|}\rho_{\mathrm{col}(i)}), \tag{4.2}$$

where $|i|$ is determined by $\mathfrak{b}$ as in (2.1).

One may check that

$$\begin{aligned}[\tilde{e}_{i,j}, \tilde{e}_{h,k}] = (\tilde{e}_{i,k} - \delta_{ik}(-1)^{|i|}\rho_{\mathrm{col}(i)})\delta_{hj} \\ - (-1)^{(|i|+|j|)(|h|+|k|)}\delta_{i,k}(\tilde{e}_{h,j} - \delta_{hj}(-1)^{|j|}\rho_{\mathrm{col}(j)}).\end{aligned} \tag{4.3}$$

Also, for any $1 \leq i,j \leq m+n$, we have

$$\chi(\tilde{e}_{i,j}) = \begin{cases} (-1)^{|i|+1} & \text{if } \mathrm{row}(i) = \mathrm{row}(j) \text{ and } \mathrm{col}(i) = \mathrm{col}(j)+1; \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

For $1 \leq i,j \leq m+n$, we let $t_{ij;\mathfrak{b}}^{(0)} := \delta_{i,j}$ and for $1 \leq r \leq l$, define

$$t_{ij;\mathfrak{b}}^{(r)} := \sum_{s=1}^{r} \sum_{\substack{i_1,\ldots,i_s \\ j_1,\ldots,j_s}} (-1)^{|i_1|+\cdots+|i_s|}\tilde{e}_{i_1,j_1}\cdots\tilde{e}_{i_s,j_s} \tag{4.5}$$

where the second sum is over all $i_1,\ldots,i_s,j_1,\ldots,j_s \in J$ such that

(1) $\deg(e_{i_1,j_1}) + \cdots + \deg(e_{i_s,j_s}) = r$;
(2) $\mathrm{col}(i_t) \leq \mathrm{col}(j_t)$ for each $t = 1,\ldots,s$;
(3) $\mathrm{col}(j_t) < \mathrm{col}(i_{t+1})$ for each $t = 1,\ldots,s-1$;
(4) $\mathrm{row}(i_1) = i$, $\mathrm{row}(j_s) = j$;
(5) $\mathrm{row}(j_t) = \mathrm{row}(i_{t+1})$ for each $t = 1,\ldots,s-1$.

First note that these elements depends on the choice of $\mathfrak{b}$. Also, the restrictions (1) and (2) imply that $t_{ij;\mathfrak{b}}^{(r)}$ belong to $\mathrm{F}_r U(\mathfrak{p})$ with respect to the Kazhdan grading. Define the following series for all $1 \leq i,j \leq m+n$:

$$t_{ij;\mathfrak{b}}(u) := \sum_{r \geq 0} t_{ij;\mathfrak{b}}^{(r)} u^{-r} \in U(\mathfrak{p})[[u^{-1}]]. \tag{4.6}$$

Let $T(Mat_l)$ be the tensor algebra of the $l\times l$ matrices space over $\mathbb{C}$ and $\mathfrak{g}=\mathfrak{gl}_{M|N}$ where $M=ml$, $N=nl$. For all $1\leq i,j\leq m+n$, define a $\mathbb{C}$-linear map $t_{ij;\mathfrak{b}}:T(Mat_l)\to U(\mathfrak{g})$ inductively by

$$t_{ij;\mathfrak{b}}(1):=\delta_{i,j},\qquad t_{ij;\mathfrak{b}}(e_{a,b}):=(-1)^{|i|}e_{i\star a,j\star b},$$
$$t_{ij;\mathfrak{b}}(x_1\otimes x_2\otimes\dots x_r):=\sum_{1\leq i_1,i_2,\dots,i_{r-1}\leq m+n}t_{ii_1;\mathfrak{b}}(x_1)t_{i_1i_2;\mathfrak{b}}(x_2)\cdots t_{i_{r-1}j;\mathfrak{b}}(x_r),\tag{4.7}$$

for $1\leq a,b\leq l$, $r\geq 1$ and $x_1,\dots,x_r\in Mat_l$, where $i\star a$ is defined to be the number in the (i,a)-th position of π, where we label the rows and columns from top to bottom and from left to right. For example, let π be the rectangle in (3.2), then $t_{23;\mathfrak{b}}(e_{2,4})=(-1)^{|2|}e_{2\star 2,3\star 4}=-e_{4,\overline{8}}$.

One may check by induction on r that for each fixed $\mathfrak{b}$,

$$\begin{aligned}&[t_{ij;\mathfrak{b}}(x),t_{hk;\mathfrak{b}}(y_1\otimes\cdots\otimes y_r)]=\\&\qquad(-1)^{|i|\,|j|+|i|\,|h|+|j|\,|h|}\Big(\sum_{s=1}^{r}t_{hj;\mathfrak{b}}(y_1\otimes\cdots\otimes y_{s-1})t_{ik;\mathfrak{b}}(xy_s\otimes\cdots\otimes y_r)\\&\qquad\qquad-t_{hj;\mathfrak{b}}(y_1\otimes\cdots\otimes y_sx)t_{ik;\mathfrak{b}}(y_{s+1}\otimes\cdots\otimes y_r)\Big),\end{aligned}\tag{4.8}$$

where the products xy_s and y_sx on the right are the matrix products in Mat_l.

For an indeterminate u, we extend the scalars from $\mathbb{C}$ to $\mathbb{C}[u]$ in the obvious way. Introducing the following matrix $A(u)$ with entries in the algebra $T(Mat_l)[u]$:

$$A(u)=\begin{pmatrix}u+e_{1,1}+\rho_1 & e_{1,2} & e_{1,3} & \cdots & e_{1,l}\\ 1 & u+e_{2,2}+\rho_2 & e_{2,3} & & e_{2,l}\\ 0 & 1 & \ddots & & \vdots\\ \vdots & & 1 & u+e_{l-1,l-1}+\rho_{l-1} & e_{l-1,l}\\ 0 & \cdots & 0 & 1 & u+e_{l,l}+\rho_l\end{pmatrix}.$$

A key observation is that for all $1\leq i,j\leq m+n$ and $0\leq r\leq l$, the element $t_{ij;\mathfrak{b}}^{(r)}\in U(\mathfrak{p})$ defined by (4.5) equals to the u^{l-r}-coefficient of $t_{ij;\mathfrak{b}}\big(\operatorname{rdet}A(u)\big)$, where

$$\operatorname{rdet}A:=\sum_{\tau\in S_l}\operatorname{sgn}(\tau)a_{1,\tau(1)}a_{2,\tau(2)}\cdots a_{l,\tau(l)},$$

for a matrix $A=(a_{i,j})_{1\leq i,j\leq l}$. We also let $A_{p,q}(u)$ stand for the submatrix of $A(u)$ consisting only of rows and columns numbered by $p,\dots,q$.

Proposition 4.1. *For all $1\leq i,j\leq m+n$, $0\leq r\leq l$ and a fixed $\mathfrak{b}$, the elements $t_{ij;\mathfrak{b}}^{(r)}$ of $U(\mathfrak{p})$ are $\mathfrak{m}$-invariant under the χ-twisted action.*

Proof. Firstly we observe that when $l\leq 1$ the results are trivial, hence we assume $l\geq 2$. Note that $\mathfrak{m}$ is generated by elements of the form $t_{ij;\mathfrak{b}}(e_{c+1,c})$, hence it suffices to show that

$$\operatorname{pr}_\chi\left([t_{ij;\mathfrak{b}}(e_{c+1,c}),t_{hk;\mathfrak{b}}(\operatorname{rdet}A(u))]\right)=0$$

for all $1\leq i,j,h,k\leq m+n$ and $1\leq c\leq l-1$. In this proof, we omit the fixed $\mathfrak{b}$ in our notation which shall cause no confusion.

By (4.8), up to an irrelevant sign, we have

$$[t_{ij}(e_{c+1,c}), t_{hk}(\operatorname{rdet} A(u))] =$$
$$t_{hj}(\operatorname{rdet} A_{1,c-1}(u))t_{ik}(\operatorname{rdet} \begin{pmatrix} e_{c+1,c} & e_{c+1,c+1} & \cdots & e_{c+1,l} \\ 1 & u+e_{c+1,c+1}+\rho_{c+1} & \cdots & e_{c+1,l} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & u+e_{l,l}+\rho_l \end{pmatrix})$$
$$- t_{hj}(\operatorname{rdet} \begin{pmatrix} u+e_{1,1}+\rho_1 & \cdots & e_{1,c} & e_{1,c} \\ 1 & \ddots & & \vdots \\ \vdots & & u+e_{c,c}+\rho_c & e_{c,c} \\ 0 & \cdots & 1 & e_{c+1,c} \end{pmatrix})t_{ik}(\operatorname{rdet} A_{c+2,l}(u)).$$

A crucial observation, which can be deduced from (4.3) and (4.4), is that for $1 \le i, j \le m+n$ and $1 \le c \le l-1$, we have

$$t_{ij}\big(e_{c+1,c}(u+e_{c+1,c+1}+\rho_{c+1})\big) \equiv t_{ij}\big(u+e_{c+1,c+1}+\rho_c\big) \pmod{I_\chi}.$$

Therefore,

$$[t_{ij}(e_{c+1,c}), t_{hk}(\operatorname{rdet} A(u))] \equiv$$
$$t_{hj}(\operatorname{rdet} A_{1,c-1}(u))t_{ik}(\operatorname{rdet} \begin{pmatrix} 1 & e_{c+1,c+1} & \cdots & e_{c+1,l} \\ 1 & u+e_{c+1,c+1}+\rho_c & \cdots & e_{c+1,l} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & u+e_{l,l}+\rho_l \end{pmatrix})$$
$$- t_{hj}(\operatorname{rdet} \begin{pmatrix} u+e_{1,1}+\rho_1 & \cdots & e_{1,c} & e_{1,c} \\ 1 & \ddots & & \vdots \\ \vdots & & u+e_{c,c}+\rho_c & e_{c,c} \\ 0 & \cdots & 1 & 1 \end{pmatrix})t_{ik}(\operatorname{rdet} A_{c+2,l}(u))$$

modulo I_χ. Making the obvious row and column operations gives that

$$\operatorname{rdet} \begin{pmatrix} 1 & e_{c+1,c+1} & \cdots & e_{c+1,l} \\ 1 & u+e_{c+1,c+1}+\rho_c & \cdots & e_{c+1,l} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & u+e_{l,l}+\rho_l \end{pmatrix} = (u+\rho_c)\operatorname{rdet} A_{c+2,l}(u),$$

$$\operatorname{rdet} \begin{pmatrix} u+e_{1,1}+\rho_1 & \cdots & e_{1,c} & e_{1,c} \\ 1 & \ddots & & \vdots \\ \vdots & & u+e_{c,c}+\rho_c & e_{c,c} \\ 0 & \cdots & 1 & 1 \end{pmatrix} = (u+\rho_c)\operatorname{rdet} A_{1,c-1}(u).$$

Substituting these back shows that $\operatorname{pr}_\chi\big([t_{ij}(e_{c+1,c}), t_{hk}(\operatorname{rdet} A(u))]\big) = 0.$ □

Set a new notation $t_{ij;\mathfrak{b}}(e_{r,r}) = (-1)^{|i|}e_{i,j}^{[r]} \in U(\mathfrak{h})$, where $\mathfrak{h} \cong \mathfrak{gl}_{m|n}{}^{\oplus l}$. Clearly, $\mathfrak{h}$ has a basis $\{e_{i,j}^{[r]} | 1 \leq r \leq l, 1 \leq i, j \leq m+n\}$. Define

$$\eta : U(\mathfrak{h}) \to U(\mathfrak{h}), \quad e_{i,j}^{[r]} \mapsto e_{i,j}^{[r]} - \delta_{ij}\rho_r.$$

Let $\xi : U(\mathfrak{p}) \to U(\mathfrak{h})$ be the algebra homomorphism induced from the natural projection $\mathfrak{p} \twoheadrightarrow \mathfrak{h}$. Define the map $\mu := \eta \circ \xi : U(\mathfrak{p}) \to U(\mathfrak{h}) \cong U(\mathfrak{gl}_{m|n})^{\otimes l}$.

Lemma 4.2. *For $1 \leq i, j \leq m+n$ and $r > 0$,*

$$\mu(t_{ij;\mathfrak{b}}^{(r)}) = \sum_{1 \leq s_1 < \cdots < s_r \leq l} \ \sum_{1 \leq i_1, \ldots, i_{r-1} \leq m+n} (-1)^{|i|+|i_1|+\cdots+|i_{r-1}|} e_{i,i_1}^{[s_1]} e_{i_1,i_2}^{[s_2]} \cdots e_{i_{r-1},j}^{[s_r]}.$$

Proof. Applying the map $e_{r,s} \mapsto \delta_{r,s}(e_{r,r} - \rho_r)$ to the matrix $A(u)$ gives a diagonal matrix with determinant $(u+e_{1,1})(u+e_{2,2})\cdots(u+e_{l,l})$, where its u^{l-r}-coefficient equals to $\sum_{1 \leq s_1 < \cdots < s_r \leq l} e_{s_1,s_1} e_{s_2,s_2} \cdots e_{s_r,s_r}$. Now the lemma follows from (4.7). $\square$

Now we are ready to state and prove our main result of this article.

Theorem 4.3. *Let π be an (m,n)-colored rectangle and $\mathfrak{b}$ be the ϵ-δ sequence determined by the rows of π. Then there exists an isomorphism $Y_{m|n}^{\mathfrak{b},l} \cong \mathcal{W}_\pi$ of filtered superalgebras such that the generators*

$$\{t_{ij;\mathfrak{b}}^{(r)} | 1 \leq i, j \leq m+n, 1 \leq r \leq l\}$$

of $Y_{m|n}^{\mathfrak{b},l}$ are sent to the elements of $\mathcal{W}_\pi$ with the same names defined by (4.5).

Proof. Again, the result is trivial when $l \leq 1$ so we assume that $l \geq 2$. By Proposition 2.2, the set of all supermonomials in the elements

$$\{t_{ij;\mathfrak{b}}^{(r)} | 1 \leq i, j \leq m+n, 1 \leq r \leq l\}$$

of $Y_{m|n}^{\mathfrak{b},l}$ taken in some fixed order and of total degree $\leq d$ forms a basis for $F_d Y_{m|n}^{\mathfrak{b},l}$ (with respect to the canonical filtration). Since κ_l is injective, by Corollary 3.7 we have

$$\dim \kappa_l(F_d Y_{m|n}^{\mathfrak{b},l}) = \dim F_d Y_{m|n}^{\mathfrak{b},l} = \dim F_d S(\mathfrak{g}^e),$$

where $S(\mathfrak{g}^e)$ is equipped with the Kazhdan grading.

Let X_d denote the subspace of $U(\mathfrak{p})$ spanned by all supermonomials in the elements $\{t_{ij;\mathfrak{b}}^{(r)} | 1 \leq i, j \leq m+n, 1 \leq r \leq l\}$ defined by (4.5) taken in some fixed order and of total degree $\leq d$. By Lemma 4.2 and the discussion above, we have $\mu(X_d) = \kappa_l(F_d Y_{m|n}^{\mathfrak{b},l})$.

Proposition 4.1 assures that $X_d \subseteq F_d \mathcal{W}_\pi$. Together with Proposition 3.5, we have

$$\dim F_d S(\mathfrak{g}^e) = \dim \mu(X_d) \leq \dim X_d \leq \dim F_d \mathcal{W}_\pi \leq \dim F_d S(\mathfrak{g}^e).$$

Thus equality holds everywhere and hence $X_d = F_d \mathcal{W}_\pi$. Moreover, μ is injective and $\mu(t_{ij;\mathfrak{b}}^{(r)}) = \kappa_l(t_{ij;\mathfrak{b}}^{(r)})$ for all $1 \leq i, j \leq m+n$ and $0 \leq r \leq l$. The composition $\mu^{-1} \circ \kappa_l : Y_{m|n}^{\mathfrak{b},l} \to \mathcal{W}_\pi$ gives the required isomorphism. $\square$

Fixe a $\mathfrak{b}$ and consider the following inverse system

$$Y_{m|n}^{\mathfrak{b},l} \twoheadleftarrow Y_{m|n}^{\mathfrak{b},l+1} \twoheadleftarrow Y_{m|n}^{\mathfrak{b},l+2} \twoheadleftarrow \cdots,$$

where the maps are homomorphisms of filtered superalgebras with respect to the canonical filtration. Comparing [Go, Theorem 1] and Proposition 2.2, we have that $Y_{m|n}^{\mathfrak{b}} = \varprojlim Y_{m|n}^{\mathfrak{b},l}$, where the inverse limit is taken in the category of filtered superalgebras. So we may view $Y_{m|n}^{\mathfrak{b}}$ as the limiting case $l \to \infty$ of the truncated super Yangian of level l.

Now let $\mathfrak{b}' \neq \mathfrak{b}$ be an ϵ-δ sequence and consider the inverse system as well:

$$Y_{m|n}^{\mathfrak{b}',l} \twoheadleftarrow Y_{m|n}^{\mathfrak{b}',l+1} \twoheadleftarrow Y_{m|n}^{\mathfrak{b}',l+2} \twoheadleftarrow \cdots,$$

and we let $Y_{m|n}^{\mathfrak{b}'}$ be the inverse limit. Let π' be the (m,n)-colored rectangle where its rows are colored with respect to $\mathfrak{b}'$. Note that we may obtain π' by switching the rows of π and re-enumerating the boxes.

Since Theorem 4.3 holds for any fixed $\mathfrak{b}$, we have that for each l,

$$Y_{m|n}^{\mathfrak{b},l} \cong \mathcal{W}_\pi \cong \mathcal{W}_{\pi'} \cong Y_{m|n}^{\mathfrak{b}',l} \tag{4.9}$$

as filtered superalgebras, where the isomorphism $\mathcal{W}_\pi \cong \mathcal{W}_{\pi'}$ in (4.9) follows from Remark 3.4. Therefore, we may simply use $\mathcal{W}_l$ to denote the W-superalgebra associated to any (m,n)-colored rectangle of base l.

Let $\mathfrak{p}^l$ be the nilpotent subalgebra of $\mathfrak{gl}_{ml|nl}$ defined as in (3.1) and $\mathcal{W}_l$ be the W-superalgebra associated to a rectangle π of base l. Let $\dot{\pi}$ be the rectangle obtained by eliminating the last column of π. Similarly, we define $\mathfrak{p}^{l-1}$ and $\mathcal{W}_{l-1}$ to be the corresponding subalgebras associated to $\dot{\pi}$. Define a surjective map $\Phi : \mathfrak{p}^l \twoheadrightarrow \mathfrak{p}^{l-1}$ by

$$\Phi(e_{i,j}) = \begin{cases} 0 & \text{if } i \text{ or } j \text{ belongs to the last column of } \pi, \\ e_{i,j} & \text{otherwise,} \end{cases}$$

which extends to an epimorphism $\Phi : U(\mathfrak{p}^l) \twoheadrightarrow U(\mathfrak{p}^{l-1})$.

As a result, we have the following commutative diagram

$$\begin{array}{ccccc} Y_{m|n}^{\mathfrak{b},l} & \xrightarrow{\cong} & \mathcal{W}_\pi \cong \mathcal{W}_l \cong \mathcal{W}_{\pi'} & \xleftarrow{\cong} & Y_{m|n}^{\mathfrak{b}',l} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle q'} \\ Y_{m|n}^{\mathfrak{b},l-1} & \xrightarrow{\cong} & \mathcal{W}_{\dot{\pi}} \cong \mathcal{W}_{l-1} \cong \mathcal{W}_{\dot{\pi}'} & \xleftarrow{\cong} & Y_{m|n}^{\mathfrak{b}',l-1} \end{array}$$

where the horizontal isomorphisms are obtained from Theorem 4.3 and (4.9), q and q' are natural projections in the inverse systems, and the map ϕ is the restriction of Φ to the subspace $\mathcal{W}_l$ of $U(\mathfrak{p}^l)$. By this commuting diagram, the following corollary is established.

Corollary 4.4. *The super Yangians $Y_{m|n}^{\mathfrak{b}}$ are isomorphic for every $\mathfrak{b}$.*

Remark 4.5. Note that the proof of Corollary 4.4 is not constructive and an explicit description of the isomorphism between $Y_{m|n}^{\mathfrak{b}}$ and $Y_{m|n}^{\mathfrak{b}'}$ is missing except some special examples. For instance, the isomorphism between $Y_{m|m}^{\mathfrak{b}}$ and $Y_{m|m}^{\mathfrak{b}'}$ is

given by $t_{i,j}^{(r)} \mapsto (-1)^r t_{i,j}^{(r)}$, where $\mathfrak{b} = \overbrace{\delta\cdots\delta}^{m}\overbrace{\epsilon\cdots\epsilon}^{m}$ and $\mathfrak{b}' = \overbrace{\epsilon\cdots\epsilon}^{m}\overbrace{\delta\cdots\delta}^{m}$. This can be observed from (4.5), but in general the explicit isomorphisms are still unknown.

References

[BBG] J. Brown, J. Brundan and S. Goodwin, Principal W-algebras for GL$(m|n)$, arXiv:math/1205.0992

[BK] J. Brundan and A. Kleshchev, Shifted Yangians and finite W-algebras, *Advances Math.* **200** (2006), 136-195.

[BR] C. Briot and E. Ragoucy, W-superalgebras as truncations of super-Yangians, *J. Phys. A* **36** (2003), 1057-1081

[CW] S.-J. Cheng and W. Wang, *Dualities and Representations of Lie Superalgebras*, Graduate Studies in Mathematics **144**, AMS, 2013.

[Go] L. Gow, Gauss Decomposition of the Yangian $Y(\mathfrak{gl}_{m|n})$, *Comm. Math. Phys.* **276** (2007), 799-825.

[Ho] C. Hoyt, Good gradings of basic Lie superalgebras, *Israel Journal of Mathematics* **192** (2012), 251-280.

[Na] M. Nazarov, Quantum Berezinian and the classical Capelli identity, *Lett. Math. Phys.* **21** (1991), 123–131.

[Pe] Y. Peng, Parabolic presentations of the super Yangian $Y(\mathfrak{gl}_{M|N})$, *Comm. Math. Phys.* **307**, (2011), 229-259

[RS] E. Ragoucy and P. Sorba, Yangian realizations from finite W-algebras, *Comm. Math. Phys.* **203** (1999), 551-572

[Wa] W. Wang, Nilpotent orbits and finite W-algebras, *Fields Institute Communications Series* **59** (2011), 71–105.

[Zh] L. Zhao, Finite W-superalgebras for queer Lie superalgebras and higher Sergeev duality, arXiv:math/1012.2326v2

Institute of Mathematics, Academia Sinica, Taipei City, Taiwan, 10617

E-mail address: `ynp@math.sinica.edu.tw`